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Levinson's theorem for a fermion–monopole system

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Abstract. We obtain an expression for Levinson's theorem for a charged fermion moving in the background field of a Dirac monopole.

1. Introduction

One of the gems of non-relativistic potential scattering theory is Levinson's theorem (Levinson 1949) which relates, for each partial wave, the scattering phase shifts at zero energy to the number of bound states. The physical insights afforded by this result are of value in our understanding of collision theory. Surprisingly, however, it was many years later that its extension to Dirac particles was obtained (Barthélémy 1967). Recently several versions of Levinson's theorem were given for charged Dirac particles moving in a background monopole field (Grossman 1983a, b, Yamagishi 1983, Ma 1985). The motion of a charged fermion in a monopole field is extremely interesting because of its novel features (for instance, the existence of a θ vacuum and zero-energy bound solution). Moreover, its generalisation to non-Abelian monopoles is much in vogue today. Discussions of the bound states for the charged fermion-monopole system are of great interest in their own right (Kazama and Yang 1977). In this paper we obtain a new form of Levinson's theorem for a charged fermion in a background Dirac monopole field following a procedure advocated by Ma and Ni (1985). We relate the phase shifts at zero momentum to the number of bound states for the lowest wave. Although our treatment will be similar to Ma's (1985) ours differs from his in that we consider here a fermion moving in the field of a Dirac monopole for which a boundary condition at the origin is needed and which has a bound state depending on this boundary condition. In Ma's treatment a background SU(5) monopole field is considered which does not require a boundary condition at the origin; moreover, he does not consider bound states in his discussion.

In the discussion to follow it will be very useful to assume the existence of a short-range radial potential $V(r)$ about the monopole. In fact, for a grand unified monopole its structure up to about 10^{-11} cm is complicated and it is only outside this range that the monopole magnetic field is dominant (Preskill 1984). This assumption might be a useful substitute for our lack of knowledge about the monopole core. Other details of V will be left unspecified.

2. Fermion–monopole system

In this section we review some aspects of the fermion-monopole system.

We consider only the lowest partial wave $j=|q|-\frac{1}{2}$, where $q=eg=\frac{1}{2}\times$ integer and e, g are the electric and magnetic charges. Then for a charged fermion of mass M

interacting with a point (Dirac) monopole the Hamiltonian is given by (Kazama *et al* 1977, Rossi 1977)

$$H = -\mathrm{i}\frac{q}{|q|}\gamma_5\frac{\mathrm{d}}{\mathrm{d}r} - \beta M + V(r)$$
$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

and the wavefunction ψ has the form

$$\psi = \frac{1}{r}\chi \exp(-\mathrm{i}Et) = \frac{1}{r}\begin{pmatrix} F(r)\eta_m \\ G(r)\eta_m \end{pmatrix} \exp(-\mathrm{i}Et) \tag{2}$$

where η_m is the relevant monopole harmonic (Wu and Yang 1976). For simplicity we assume $q > 0$. $V(r)$ is a short-range potential; its details will be left unspecified.

The $V(r) = 0$ case has been explicitly examined by Grossman (1983a) and Yamagishi (1983). This case merits our consideration. As the Hamiltonian is not self-adjoint a boundary condition is imposed at the origin. For a recent discussion, see Roy (1985). If this condition is parametrised by an angle θ, i.e.

$$F(0)/G(0) \equiv \mathrm{i}\tan(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi) \tag{3}$$

we find that the scattering solutions are given by (Grossman 1983a, Yamagishi 1983)

$$r\psi^0_{k,\theta} = \begin{cases} [k/\pi(E - M\sin\theta)]^{1/2}\chi_{k,\theta} & E > M \\ [k/\pi(|E| + M\sin\theta)]^{1/2}\bar{\chi}_{k,\theta} & E < -M \end{cases} \tag{4}$$

where

$$\chi_{k,\theta} = \begin{pmatrix} \dfrac{\mathrm{i}k}{E-M}\cos\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)\sin kr + \mathrm{i}\sin\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)\cos kr \\ \cos\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)\cos kr - \dfrac{k}{E+M}\sin\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)\sin kr \end{pmatrix} \qquad E > M$$

$$\bar{\chi}_{k,\theta} = \begin{pmatrix} -\dfrac{\mathrm{i}k}{|E|+M}\cos\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)\sin kr + \mathrm{i}\sin\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)\cos kr \\ \cos\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)\cos kr - \dfrac{k}{-|E|+M}\sin\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)\sin kr \end{pmatrix} \qquad E < -M.$$

Here $k > 0$, and the superscript 0 refers to the $V = 0$ case. An overhead bar will be used to designate negative-energy state quantities.

The inner product is defined by

$$(\chi_1, \chi_2) = \int_0^\infty \mathrm{d}r\, \chi_1^\dagger(r)\chi_2(r)$$

and we can verify that the ψ^0 are orthogonal:

$$(\psi^0_{k,\theta}, \psi^0_{k',\theta}) = \delta(E - E'). \tag{5}$$

There is also a bound state for $\cos\theta < 0$, namely

$$B^0_\theta = \begin{pmatrix} \mathrm{i}\sin(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi) \\ \cos(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi) \end{pmatrix} \mathrm{e}^{-Kr}(2K)^{1/2}$$
$$K = M|\cos\theta| \tag{6}$$
$$(B^0_\theta, B^0_\theta) = 1.$$

Completeness is then given (where $\theta(x)$ is the step function) by

$$\tfrac{1}{2}\int \mathrm{d}E(\psi^{0\dagger}_{k,\theta}(r')\psi^{0}_{k,\theta}(r))+\tfrac{1}{2}\theta(-\cos\theta)B^{0\dagger}_{\theta}(r')B^{0}_{\theta}(r)=\delta(r-r') \tag{7}$$

where in the integration we sum over positive- and negative-energy states. Note the inclusion of the bound state in (7). We now consider the $V\neq 0$ case.

For the Hamiltonian (1) we assume that $V(r)$ vanishes beyond some distance R from the monopole. Outside this range the solutions are essentially those given above except for some phase shift. We will write these solutions in the form (for $r>R$)

$$\psi_{k,\theta,\alpha}(r)=\begin{cases}(\cos\alpha\chi_{k,\theta}+\sin\alpha\chi_{k,\theta+\pi})\{k/\pi[E-M\sin(\theta+2\alpha)]\}^{1/2} & E>M\\ (\cos\bar{\chi}_{k,\theta}+\sin\alpha\bar{\chi}_{k,\theta+\pi})\{k/\pi[|E|+M\sin(\theta+2\alpha)]\}^{1/2} & E<-M\end{cases} \tag{8}$$

where $\chi_{k,\theta}$ and $\bar{\chi}_{k,\theta}$ were defined in equation (4). Again we can verify that

$$(\psi_{k,\theta,\alpha},\psi_{k',\theta,\alpha})=\delta(E-E') \tag{9}$$

so that $\psi_{k,\theta,\alpha}$ form an orthogonal set, but, as above, they do not in general form a complete set because of the possible bound states.

From equations (4) and (8) we have, for $E>M$,

$$r\psi^{0}_{k,\theta}=\begin{pmatrix}\mathrm{i}\left(\dfrac{E+M}{\pi k}\right)^{1/2}\cos(kr+\eta^{0})\\ -\left(\dfrac{E-M}{\pi k}\right)^{1/2}\sin(kr+\eta^{0})\end{pmatrix}\qquad (r>R) \tag{10a}$$

$$r\psi_{k,\theta,\alpha}=\begin{pmatrix}\mathrm{i}\left(\dfrac{E+M}{\pi k}\right)^{1/2}\cos(kr+\eta)\\ -\left(\dfrac{E-M}{\pi k}\right)^{1/2}\sin(kr+\eta)\end{pmatrix}\qquad (r>R) \tag{10b}$$

where

$$\begin{aligned}\tan\eta^{0}&=-\frac{k}{E-M}\cot\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\\ \tan\eta&=-\frac{k}{E-M}\cot\left(\frac{\theta}{2}+\frac{\pi}{4}+\alpha\right)\end{aligned} \tag{11}$$

and, as above, the superscript 0 denotes the $V=0$ case. The phase shift δ *relative* to pure monopole scattering ($V=0$) may be defined by

$$\delta=\eta-\eta^{0}. \tag{12}$$

For the negative-energy states we replace $(E\pm M)^{1/2}$ in equation (10) by $\pm(|E|\mp M)^{1/2}$ and the phases are given by

$$\begin{aligned}\tan\bar{\eta}^{0}&=\frac{k}{|E|+M}\cot\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\\ \tan\bar{\eta}&=\frac{k}{|E|+M}\cot\left(\frac{\theta}{2}+\frac{\pi}{4}+\alpha\right).\end{aligned} \tag{13}$$

The overbars denote as usual the negative-energy quantities. At the origin the wavefunction satisfies

$$F(0)/G(0)=\mathrm{i}\tan(\tfrac{1}{2}\theta+\tfrac{1}{4}\pi+\alpha). \tag{14}$$

We recover the free-particle result (13) when α is zero. To obtain Levinson's theorem we introduce first the Green function.

3. Levinson's theorem

The Green function $G(r, r'; E)$ for (1) is defined by

$$\gamma_0(E-H)G(r, r'; E)=\delta(r-r')$$

with (here $\bar{\psi}=\psi^{+}\gamma_0$)

$$G(r, r'; E)=\sum_k \frac{\psi_{k,\theta,\alpha}(r)\bar{\psi}_{k,\theta,\alpha}(r')}{E-E_k+\mathrm{i}\varepsilon} \tag{15}$$

where the sum over k is an integration over the continuum spectrum and a sum over discrete states. For $V=0$ similar results hold with ψ replaced by ψ^0, H by H_0 and G by G^0. We also note that

$$G(r, r'; E)=G^0(r, r'; E)+\int \mathrm{d}r''\, G^0(r,r''; E)\gamma_0 VG(r'', r; E). \tag{16}$$

Following Ma and Ni (1985) we calculate the quantity

$$-\frac{1}{\pi}\operatorname{Im}\int \mathrm{d}E\int \mathrm{d}r\, \mathrm{Tr}\{\gamma_0[G(r, r; E)-G^0(r, r; E)]\}$$

$$=-\frac{1}{\pi}\operatorname{Im}\int \mathrm{d}E\int \mathrm{d}r\int \mathrm{d}r'\, \mathrm{Tr}\{\gamma_0 G_0(r, r'; E)\gamma_0 V(r')G(r', r; E)\}$$

$$=-\frac{1}{\pi}\operatorname{Im}\int \mathrm{d}E\sum_{k,k'}\frac{\langle\psi_{k'}|\psi_k^0\rangle\langle\psi_k^0|V|\psi_{k'}\rangle}{(E-E_k+\mathrm{i}\varepsilon)(E-E_{k'}+\mathrm{i}\varepsilon')}$$

where we have suppressed the subscripts θ, α. Using

$$\operatorname{Im}\frac{E_{k'}-E_k}{(E-E_k+\mathrm{i}\varepsilon)(E-E_{k'}+\mathrm{i}\varepsilon')}=-\pi[\delta(E-E_{k'})-\delta(E-E_k)]$$

$$\langle\psi_k^0|V|\psi_{k'}\rangle=\int \mathrm{d}r\,\psi_k^{0+}(H-H_0)\psi_{k'}=(E_{k'}-E_k)\langle\psi_{k'}|\psi_k^0\rangle^*$$

we find that

$$-\frac{1}{\pi}\operatorname{Im}\int \mathrm{d}E\int \mathrm{d}r\, \mathrm{Tr}\{\gamma_0[G(r, r; E)-G^0(r, r; E)]\}$$

$$=\int \mathrm{d}E\sum_{k,k'}[\delta(E-E_{k'})-\delta(E-E_k)]\langle\psi_{k',\theta,\alpha}|\psi_{k,\theta}^0\rangle\langle\psi_{k,\theta}^0|\psi_{k',\theta,\alpha}\rangle. \tag{17}$$

When the integration range of E in (17) is from $-\infty$ to $+\infty$, the δ cancel and the integral vanishes. If the integration range is from $-M$ to $+M$, $\delta(E-E_k)=1$ for the bound solution (6). Thus if $\cos\theta\geqslant 0$ and $-M<E<M$, we have for the right-hand side of (17)

$$\sum_{k,k'}\langle\psi_{k',\theta,\alpha}|\psi_{k,\theta}^0\rangle\langle\psi_{k,\theta}^0|\psi_{k',\theta,\alpha}\rangle=\sum_{k'}\langle\psi_{k',\theta,\alpha}|\psi_{k',\theta,\alpha}\rangle=N \tag{18}$$

where N is the number of bound states. However, if $\cos\theta > 0$ then the right-hand side of (17) is $N-1$, instead.

The left-hand side of (17) may be evaluated directly by making use of

$$-\frac{1}{\pi}\,\mathrm{Im}\int \mathrm{d}r\,\mathrm{Tr}[\gamma_0 G(r,r;E)] = \sum_k \delta(E-E_k)\lim_{E_{k'}\to E_k}\langle\psi_{k,\theta,\alpha}|\psi_{k',\theta,\alpha}\rangle \tag{19}$$

and a similar one for G^0. Moreover, since the wavefunction $\psi_{k,\theta,\alpha}$ obeys the Dirac equations

$$\begin{aligned} &-\mathrm{i}\gamma_5(\mathrm{d}/\mathrm{d}r)\psi_{k,\theta,\alpha} + \beta M\psi_{k,\theta,\alpha} + V\psi_{k,\theta,\alpha} = E_k\psi_{k,\theta,\alpha} \\ &\mathrm{i}((\mathrm{d}/\mathrm{d}r)\psi^{\dagger}_{k',\theta,\alpha})\gamma_5 + \psi^{\dagger}_{k',\theta,\alpha}\beta M + V\psi^{\dagger}_{k',\theta,\alpha} = E_{k'}\psi^{\dagger}_{k',\theta,\alpha} \end{aligned} \tag{20}$$

we have

$$\langle\psi_{k,\theta,\alpha}|\psi_{k',\theta,\alpha}\rangle = \lim_{r\to\infty}\frac{\mathrm{i}}{E_{k'}-E_k}(F^*_{k,\theta,\alpha}G_{k',\theta,\alpha} + G^*_{k,\theta,\alpha}F_{k',\theta,\alpha}). \tag{21}$$

F and G are the top and bottom elements of (10a) and (10b), respectively, depending on whether we are considering the $V=0$ or $V\neq 0$ case. The end result is

$$\begin{aligned} &\frac{1}{\pi}\left(\int_{-\infty}^{-M} + \int_{M}^{\infty}\right)\mathrm{d}E\lim_{E_{k'}-E_k}\left(\frac{\mathrm{i}}{E_{k'}-E_k}\right)\{F^*_{k,\theta,\alpha}G_{k',\theta,\alpha} \\ &\qquad + G^*_{k,\theta,\alpha}F_{k',\theta,\alpha} - F^{0*}_{k,\theta}G^0_{k',\theta} - G^{0*}_{k,\theta}F_{k',\theta}\} \\ &\qquad = \begin{cases} N & \text{if} \quad \cos\theta \geqslant 0 \\ N-1 & \text{if} \quad \cos\theta > 0 \end{cases}. \end{aligned} \tag{22}$$

In terms of the wavefunctions (10) the expression in braces at the left-hand side of (22) may be written

$$\begin{aligned} &[4\mathrm{i}/2\pi(kk')^{1/2}]\{(|E|/E)\,B_+\cos[(k'-k)r + \tfrac{1}{2}(\eta'-\eta+\eta^{0'}-\eta^0)]\sin\tfrac{1}{2}(\eta'-\eta-\eta^{0'}+\eta^0) \\ &\qquad + B_-\cos[(k'+k)r + \tfrac{1}{2}(\eta'+\eta+\eta^{0'}+\eta^0)]\sin\tfrac{1}{2}(\eta^{0'}+\eta^0-\eta'-\eta)\} \end{aligned}$$

where

$$B_\pm \equiv \tfrac{1}{2}\{[(|E|-M)(|E'|+M)]^{1/2} \pm [(|E|+M)(|E'|-M)]^{1/2}\}.$$

For $E<0$ the appropriate overbars should be appended.

In the limit $E_{k'}\to E_k$ we have $B_+\to k$,

$$\frac{\sin\tfrac{1}{2}(\eta'-\eta-\eta^{0'}+\eta^0)}{E_{k'}-E_k} \to \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}E}(\eta-\eta^0)$$

$$\frac{B_-}{E_{k'}-E_k} \to -\frac{M}{2k}\frac{|E|}{E}$$

and

$$\lim_{r\to\infty}\left(\frac{\sin 2\,kr}{k}\right) = \pi\frac{k}{|E|}\delta(|E|-M)$$

and $\cos kr \to 0$ as $r\to\infty$.

Equation (22) reduces to

$$-\frac{1}{\pi}\left(\int_M^{\infty}-\int_{-\infty}^{-M}\right)\frac{\mathrm{d}}{\mathrm{d}E}(\eta-\eta^0)\mathrm{d}E-\sin[\eta^0(M)+\eta(M)]\sin[\eta^0(M)-\eta(M)]$$
$$+\sin[\bar{\eta}^0(-M)+\bar{\eta}(-M)]\sin[\bar{\eta}^0(-M)-\bar{\eta}(-M)]$$
$$=\begin{cases}N & \text{if} \quad \cos\theta\geqslant 0\\ N-1 & \text{if} \quad \cos\theta<0\end{cases} \tag{23}$$

which is the form Levinson's theorem takes for the system we are considering here.

We may further simplify this result by noting that equations (11) and (13) imply that

$$\eta(+\infty)+\bar{\eta}(-\infty)=\eta^0(+\infty)+\bar{\eta}^0(-\infty)=0 \tag{24}$$

and the zero-momentum phases $\eta(M)$, $\eta^0(M)$, etc, are either $(n+\frac{1}{2})\pi$ or $n\pi$ (n integer) depending on the sign of E. Thus we find

$$\frac{1}{\pi}[\delta(M)+\delta(-M)]=\begin{cases}N & \text{if} \quad \cos\theta\geqslant 0\\ N-1 & \text{if} \quad \cos\theta<0\end{cases} \tag{25}$$

where δ is defined by equation (12).

Several remarks are in order. First, the sum of the phase shifts at $E=\pm M$ is independent of the sign and strength of V. This can be understood as follows: as the strength of $V(r)$ increases, scattering states at one energy ($+M$, say) may turn into bound states while bound states may also turn into scattering states at the other ($-M$) (Ma 1985). Second, the left-hand side of (24) carries some vestige of the θ angle which is not unexpected, but which is not reflected in previous results. Finally, if we follow Grossman (1983b) and interpret the derivative of the phase with respect to energy as a time delay, then the right-hand side of (24) (see also (23)) is just the sum of time delays off the negative- and positive-energy states, and this delay is entirely attributed to bound states of the system.

It would be interesting to relate the vacuum charge (Witten 1979) to the phase shifts.

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